Free energy from Euclidean quantum gravity

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The gravitational path integral in the saddle point approximation can give us not only the entropy, but also the full free energy $F(\beta)$:

$$F(\beta) \approx \frac{1}{\beta} I_E(g_{cl})$$

but if the geometry has infinite volume, this requires regularization and infinite subtraction.

The simplest is to subtract the action of Minkowski space w/ the same boundary.

Since, on-shell, $R=0$, we have

$$\beta F = I_E(g_{cl}) - I_E^0(g_{Mink}) = -\frac{1}{8\pi G}\int_{\partial M} \sqrt{h}\,(K-K^0) \quad (\text{no } h^0)$$

In gravity, action, energy, mass etc are measured by comparing how a sphere at large distance is curved (extrinsically) compared with that sphere in flat space. In gravity, all physical magnitudes are geometrical, and G is merely a conversion factor to non-geometric quantities.

$$ds^2 = \left(1-\frac{2M}{r}\right)d\tau^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega_{(2)}$$

$$ds_0^2 = d\tau_0^2 + dr^2 + r^2 d\Omega_{(2)}$$

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The spheres are the same (because we used area-radius coordinate), but for the time to match we must have

$$\tau_0 = \sqrt{1-\frac{2M}{R}}\,\tau \simeq \left(1-\frac{M}{R}+\cdots\right)\tau$$

With this, the two boundary geometries are the same.

We must then compute their respective extrinsic curvatures.

Use that in general $\sqrt{h}\,K = n^i \partial_i \sqrt{h}$.

Then integrate over τ and over $\Omega_{(2)}$.

After a short calculation we find

$$\beta F = \frac{1}{2}\beta M = \frac{1}{16\pi}\beta^2 \qquad T = \frac{1}{8\pi M} \qquad M = \frac{\beta}{8\pi}$$

so then

$$E = \partial_\beta(\beta F) = \frac{1}{8\pi}\beta = M \quad \checkmark$$

$$S = (\beta\partial_\beta - 1)(\beta F) = \frac{1}{16\pi}\beta^2 = 4\pi M^2 = \frac{A_H}{4} \quad \checkmark$$

$$\left(\text{or } S = \beta(F-E)\right) \qquad A_H = 4\pi(2M)^2$$

So this approach reproduces the correct thermodynamics.

Observe that the specific heat is negative:

$$C = \frac{T dS}{dT} = -\beta \frac{\partial S}{\partial \beta} = -\frac{\beta^2}{8\pi} < 0$$

If put in equilibrium w/ a bath of radiation at the same temperature, $T = \frac{1}{8\pi M}$, the equilibrium will be unstable.

- in a fluctuation where the bh absorbs more than emits, its $M \nearrow$ $T \searrow$: it cools down, absorbs more
- in a fluctuation where the bh emits more than absorbs, its $M \searrow$ $T \nearrow$: it heats up, radiates even more

This is typical of bhs in asymptotically flat space, but not of black holes in "boxes".